Non-integer flux quanta for a spherical superconductor

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Abstract

A thin film superconductor shaped into a spherical shell at whose center lies the end of long thin solenoid in which there is an integer flux $N\Phi_0$ has been previously extensively studied numerically as a model of a two-dimensional superconductor. The emergent flux from the solenoid produces a radial B-field at the superconducting shell and N vortices in the superconducting film. We study here the effects of including a second solenoid (carrying a flux f) which is inserted inside the first solenoid but passing right across the sphere. This Aharonov-Bohm (AB) flux does not have to be quantized to make the order parameter single valued. The Ginzburg-Landau (GL) free energy is minimized at fixed N as a function of f and it is found that the minimum is usually achieved when the AB flux f is half a flux quantum, but depending on N the minimum may be at $f = 0$ or values which are not obvious rational fractions.

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A spherical shell with a magnetic monopole at its center provides a useful geometry for numerical studies of a thin film (two-dimensional) superconductor in a perpendicular magnetic field[1]. Furthermore, an investigation of the ground state of the vortices in this system revealed an interesting geometric effect. While the ground state of vortices penetrating an *infinite* type II superconductor plane is the well-known Abrikosov vortex lattice[2], where vortices form a triangular array, on a spherical surface a perfect triangular lattice cannot form without the presence of at least twelve disclination defects and when N, the number of vortices on the surface is large other defects appear[3].

In this paper, we descibe a situation where again geometry and topology play key roles. We consider as in Ref. 1 the ground state of vortices in a spherical superconductor in the presence of a radial magnetic field generated by a monopole at the center of the sphere, but the vector potential describing the magnetic field contains also an additional Aharonov-Bohm (AB) flux. It should be possible to realize this system experimentally by inserting a solenoid into the center of a spherical superconducting shell as the field which emerges from the end of the solenoid approximates to the field from a magnetic monopole. One may visualize this system as a spherical superconductor with a couple of very thin solenoids one of which ends at the center of the sphere and the other lies along the z-axis. (See Fig. 1.)

The quantum mechanical system which consists of a magnetic monopole *and* an AB flux is known to have many unusual properties as discussed in Ref. 4 in detail. For instance, there exist solutions to the Schrödinger equation for which the Dirac quantization of monopole charges[5] does not hold even if the wavefunction is required to be single-valued. In our system, the monopole charges should be quantized, since it is determined by the number of vortices penetrating the superconductor as discussed below. One of the main results of this paper is that, in the ground state, the system organizes itself such that a nonvanishing AB flux is induced with its strength given by a fraction of the fundamental flux quantum with the actual value related to N, the number of vortices, in a way which is obscure to us. This 'quantization' of the AB flux differs in its origin from other types of flux quantizations which are usually determined from a topological consideration such as the requirement that the

order parameter remain single valued as one passes round a circuit. In the present system, the flux is quantized *dynamically* in the sense that it is determined by minimizing the free energy for a superconductor.

We consider a spherical type II superconductor of radius R and width $d \ll R$ in the presence of a radial magnetic field $\mathbf{H} = H(r)\hat{\mathbf{r}}$. This magnetic field is produced by a magnetic monopole at the center of the sphere. The system can be described by the following Ginzburg-Landau free energy for a (complex) superconducting order parameter $\Psi(\theta, \phi)$:

$$F[\Psi, \Psi^*, \mathbf{A}] = dR^2 \int d\Omega \, [\frac{\hbar^2}{2m}|\mathbf{D}\Psi(\theta, \phi)|^2 + \alpha|\Psi|^2$$
$$+ \frac{\beta}{2}|\Psi|^4 + \frac{1}{8\pi}|\nabla \times \mathbf{A} - \mathbf{H}|^2], \tag{1}$$

where $d\Omega$ is the solid angle element, α, β and m are phenomenological parameters, and $\mathbf{D} = -i\nabla - (e^*/\hbar c)\mathbf{A}$ with the vector potential $\mathbf{A}$ and the charge of a Cooper pair $e^* = 2e$. Physical properties of the system including the effect of fluctuations are described by the partition function given by

$$Z = \int \mathcal{D}\Psi \mathcal{D}\Psi^* \mathcal{D}\mathbf{A} \, \exp(-F/k_B T). \tag{2}$$

We study the ground state of the system by employing the mean-field theory in which one approximates the partition function in (2) by the saddle point values of F, and neglects all the fluctuations. Furthermore we shall work in the extreme Type II superconducting limit. For the vector potential (or for the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$), this amounts to taking the magnetic induction such that its value is equal to the magnetic field from the monopole at the superconducting surface. In terms of the vector potential, one could take the following standard form for a magnetic monopole with charge g, which has a Dirac singularity along the negative z-axis: $A_r = A_\theta = 0$, $A_\phi = (g/r)\tan(\theta/2)$. Note that $B = g/R^2$. However, a more general form for $\mathbf{A}$ will be considered in the functional integral in (2) by adding an Aharonov-Bohm potential which describes the effects of an infinitely thin solenoid carrying flux f along the z-axis in addition to the magnetic field produced by the monopole. On the surface of the superconductor, the vector potential in this case can be written as

$$A_r = A_\theta = 0, \quad A_\phi = BR \tan\frac{\theta}{2} + \frac{f}{2\pi R \sin\theta}. \tag{3}$$

Note that except on the z-axis this vector potential gives the same $\mathbf{B}$ as in the case without the AB flux. The presence of such AB potential, however, makes a profound change in the spectrum of the operator $\mathbf{D}^2$ in a spherical geometry[4].

As for the order parameter, we employ the lowest Landau level (LLL) approximation, which one usually uses at the mean-field theory level to describe the vortex lattice[2]. We expand Ψ in terms of the eigenstates of $\mathbf{D}^2$ and discard all the higher eigenfunctions except the lowest one. For a system consisting of N vortices, the flux quantization condition, which results from the single-valuedness of the order parameter around a vortex, gives that the total flux through the surface of the sphere, $B(4\pi R^2)$ is equal to N times the fundamental flux quantum $\Phi_0 = hc/e^*$. This condition is equivalent to the Dirac's quantization condition for the monopole charge g: $2e^*g = N\hbar c$. The normalized LLL wavefunctions are given by[4]

$$\psi_m(\theta, \phi; a) = h_m e^{im\phi} \sin^{|m+a|}(\frac{\theta}{2}) \cos^{|N-m-a|}(\frac{\theta}{2}), \tag{4}$$

where an integer m labels the degeneracy of the LLL, $a = -f/\Phi_0$, and the normalization constant $h_m = [4\pi R^2 \mathcal{B}(|m+a|+1, |N-m-a|+1)]^{-1/2}$ with the beta function $\mathcal{B}$. The corresponding eigenvalues of $\mathbf{D}^2$ are $[l(l+1) - N^2/4]/R^2$, where

$$l = \frac{1}{2}[|m+a| + |N-m-a|]. \tag{5}$$

Since zeros of the order parameter are identified with vortices, we might expect that $N+1$ eigenfunctions are needed to describe the positions of N vortices and an overall constant in the order parameter. We note then that, because of the peculiar form of the eigenvalues given in (5), one has to include at least one eigenfunction which has a larger eigenvalue than the rest of the eigenfunctions. Let us first focus on the case where $0 < a < 1$. The remaining cases will be discussed later. We may take eigenfunctions with $m = 0, 1, 2 \cdots N$ (which is not a unique choice as discussed below) to form a LLL order parameter as follows:

$$\psi(\theta, \phi) = P \sum_{m=0}^{N} v_m \psi_m(\theta, \phi; a), \tag{6}$$

4

where $P = (k_B Thc/d\beta Be^*)^{1/4}$ and v_m are complex numbers. Inserting this into (1) we obtain

$$\frac{F[\{v_m\}]}{k_B T} = \alpha_T [\sum_{m=0}^{N-1} |v_m|^2 + \epsilon(a)|v_N|^2]$$
$$+ \frac{1}{2N} \sum_{k_i=0}^{N} \delta_{k_1+k_2,k_3+k_4} w_{1234} v_{k_1}^* v_{k_2}^* v_{k_3} v_{k_4}, \qquad (7)$$

where

$$\alpha_T = \frac{dP^2}{k_B T} (\alpha + \frac{\hbar e^* B}{2mc})$$

is the dimensionless temperature which changes sign at the mean field transition line, $H_{c2}(T)$,

$$\epsilon(a) = 1 + \frac{2a}{\alpha_T}(1 + \frac{1+a}{N}) \frac{dP^2}{k_B T} \frac{\hbar e^* B}{2mc}, \qquad (8)$$

and

$$w_{1234} = \frac{\mathcal{B}(\frac{1}{2}\sum_i k_i + 2a + 1, \frac{1}{2}\sum_i |N - k_i - a| + 1)}{\prod_i [\mathcal{B}(k_i + a + 1, |N - k_i - a| + 1)]^{1/2}} \qquad (9)$$

Finally, if we rescale $v_m = |\alpha_T|^{1/2} u_m$, we have

$$\frac{F[\{u_m\}]}{k_B T} = \alpha_T^2 [- \sum_{m=0}^{N-1} |u_m|^2 - \epsilon(a)|u_N|^2$$
$$+ \frac{1}{2N} \sum_{k_i=0}^{N} \delta_{k_1+k_2,k_3+k_4} w_{1234} u_{k_1}^* u_{k_2}^* u_{k_3} u_{k_4}], \qquad (10)$$

where we are mainly concerned with the region below the mean field transition line (i.e. $\alpha_T < 0$) as the minus signs on the right hand side indicate. Within mean field theory the vortex lattice configuration is determined by minimizing the free energy (10) for given N. We denote the minimum free energy per vortex ($F/Nk_B T\alpha_T^2$) for given a and N by $E_N(a)$. Once the coefficients $\{u_m\}$ that minimizes the free energy are obtained, the positions of the vortices can be determined as elaborated later.

We first discuss an interesting symmetry relation in this system. We note that in (6) one could also use ψ_m with $m = -1, 0, 1 \cdots N - 1$ for which one eigenfunction (with $m = -1$) has a larger eigenvalue than the rest. If we start from

5

$$\psi(\theta, \phi) = P \sum_{m=-1}^{N-1} \tilde{v}_m \psi_m(\theta, \phi; a), \tag{11}$$

one can easily show that

$$\psi(\theta, \phi) = e^{i(N-1)\phi} P \sum_{m=0}^{N} v_m \psi_m(\pi - \theta, -\phi; 1 - a), \tag{12}$$

where $v_m \equiv \tilde{v}_{N-m-1}$. Inserting this into (1) we find that the free energy expression is just the same as (7) except that a is replaced by $1 - a$. Therefore the minimum free energy $E_N(a)$ as a function of a has a reflection symmetry about $a = 1/2$ for $0 < a < 1$:

$$E_N(1 - a) = E_N(a). \tag{13}$$

Eq. (12) also suggests that the vortex configuration at $1 - a$ can be obtained by performing the transformations; $\theta \to \pi - \theta, \phi \to -\phi$ on the vortex lattice obtained at a.

There is also a periodicity relation with respect to a in the following sense. For $1 < a < 2$ we have two choices: either $m = -1, 0, 1 \cdots N - 1$ or $m = -2, -1, 0 \cdots N - 2$. If we take the first and start from (11), we can show that

$$\psi(\theta, \phi) = e^{-i\phi} P \sum_{m=0}^{N} v_m \psi_m(\theta, \phi; a - 1), \tag{14}$$

where $v_m \equiv \tilde{v}_{m-1}$ in this case. Proceeding as in the previous case, we conclude that

$$E_N(a) = E_N(a - 1), \tag{15}$$

and the vortex lattice has the same configuration for both cases. One can easily see that (15) holds for all a (including integral values of a). Together with (13) it characterizes the ground state energy spectrum as a function of the flux strength. Keeping these relations in mind, we will restrict our discussion below only to the region $0 \le a \le \frac{1}{2}$.

We study the ground state of the vortex system described by (1) and (3) by numerical minimization of (10). We use a straightforward quasi-Newtonian algorithm. The free energy (10) is invariant under the transformations, $u_m \to \exp(i\gamma)u_m$, and $u_m \to \exp(im\gamma)u_m$, for arbitrary real γ, where the latter amounts to the rotation of the vortices by an angle γ

around the z-axis. These symmetries enable one to set, for example, $\mathrm{Im} u_0 = \mathrm{Im} u_1 = 0$ to reduce the number of independent variables from $2N + 2$ ($N + 1$ complex variables) to $2N$.

The first thing we notice in the numerical minimization is that the minimum free energy occurs when $v_N = 0$, *i.e.*, the mode that gives the larger eigenvalue than the rest is not present after all. This means that, for $0 \le a < 1$, we can write the order parameter as

$$
\begin{aligned}
\psi(\theta, \phi) &= P \sin^a(\frac{\theta}{2}) \cos^{1-a}(\frac{\theta}{2}) \\
&\times \sum_{m=0}^{N-1} v_m h_m e^{im\phi} \sin^m(\frac{\theta}{2}) \cos^{N-1-m}(\frac{\theta}{2}) \\
&= C \sin^a(\frac{\theta}{2}) \cos^{1-a}(\frac{\theta}{2}) \\
&\times \prod_{k=1}^{N-1} \left(a_k \sin(\frac{\theta}{2}) e^{i\phi/2} - b_k \cos(\frac{\theta}{2}) e^{-i\phi/2} \right),
\end{aligned}
\tag{16}
$$

where $a_k = \cos(\theta_k/2) \, \exp(-i\phi_k/2)$ and $b_k = \sin(\theta_k/2) \, \exp(i\phi_k/2)$ with $(\theta, \phi) = (\theta_k, \phi_k)$ being the positions of vortices. The overall complex constant C and $N - 1$ pairs of real numbers (θ_k, ϕ_k) account for the original N complex coefficients v_m. Note that the presence of the AB flux drives the vortices into a peculiar configuration described by (16), where in addition to $N - 1$ vortices at (θ_k, ϕ_k) there are *fractional* vortices at the north ($\theta = 0$) and south ($\theta = \pi$) poles in the sense that the zeros are of order a and $1 - a$ respectively. In the limit where $a \to 0$ we obtain a configuration where one *full* vortex at the south pole. As already noticed in the previous studies[1] for $a = 0$, one is free to fix the position of one vortex, since the free energy depends only on the relative positions of vortices. Therefore the continuation from the nonzero a to $a = 0$ uniquely picks the positions of vortices among the degenerate configurations. When one continues the order parameter to $a = 1$, one reaches the configuration where one *full* vortex is on the north pole. This configuration is related to the one for $a = 0$ via (12), and is among the many degenerate ground states of vortices without an AB flux.

We obtain the ground state energy $E_N(a)$ as a function of a for various values of N. As discussed above, it is sufficient to consider the interval, $0 \le a \le \frac{1}{2}$. The main results of our investigation is that for most of the values of N we studied the ground state energy $E_N(a)$

has a minimum at nonzero values of a. This means that the vortices organize themselves such that a flux of strength $f = -a\Phi_0$ is induced along the z-axis. Typical examples of energy spectra are shown in Fig. 2. For $N = 1, 2, \cdots 36$, we have found two exceptional cases at $N = 14$ and $N = 22$ for which the energy spectrum also shows a local minimum or maximum as shown in Fig. 3. While we have not made a systematic investigation of the scale of the variation of $E_N(a)$ with AB flux f as N increases, we suspect that it is largely N independent. If so then the overall free energy change scales as N, which for large N could dwarf the magnetic energy stored in the AB solenoid and justify our use of the word "induced". In other words the AB flux in this limit could be regarded as being spontaneously generated.

For some values of N, the ground state energy $E_N(a)$ has a minimum at $a = 0$ i.e at vanishing AB flux. These cases include $N = 2, 3, 6, 12, 32, 42 \cdots$. We note that except for the first three, these numbers correspond to the so-called magic numbers found in previous studies on the vortex system on a sphere[1] without the AB flux. It was found there that at these numbers the ground state energy has a lower value than for nearby values of N, and the vortex configuration in this ground state possesses a five-fold symmetry.

The nonzero value of the induced AB flux in most cases is equal to one half of the fundamental flux quantum $(a = \frac{1}{2})$. In particular, for all odd values of N up to $N = 35$, the free energy is minimized for $a = \frac{1}{2}$. But, for some values of N, the value of a is given by interesting fractions other than $\frac{1}{2}$. These cases up to $N = 36$ are tabulated in Table. I.

The actual ground state configuration of vortices when the AB flux is present is also of interest. We obtain the configuration of vortices that minimizes the free energy for given value of a by numerically finding the roots (θ_k, ϕ_k) from the coefficients $\{u_m\}$. We study how it changes when the flux strength a is increased. As remarked above, for $a \neq 0$ there are *fractional* vortices at the north and south poles with strength a and $1 - a$ respectively. We find in general that, since the strength of the *fractional* vortex at the north pole becomes increased as the value of a increases, the nearby vortices are pushed downwards. We expect that for most cases one reaches a kind of balanced configuration between vortices on the

northern and southern hemispheres as one approaches $a = \frac{1}{2}$, which contributes to lowering the energy. An example of this behavior can be seen in Fig. 4. A word of caution should be noted at this stage, however, that the movement of vortices as the value of a is changed is not entirely a well-defined concept, since the vortex configuration is only determined up to certain symmetry relations: A rotation around the z-axis by any amount or a reflection in the x-z plane (which corresponds to $u_m \rightarrow u_m^*$) does not change the energy. In Fig. 4, for instance, in order to compare the vortex configuration for two different values of a, we have rotated the vortices until the three on the equator in both cases coincide.

Even if one fixes the positions vortices in an appropriate way as in Fig. 4, the detailed movement of vortices as the AB flux strength is increased is not a simple downward translation but more complicated. This is especially true for the case where the minimum energy is achieved for nonzero fractions other than $\frac{1}{2}$ as listed in Table I. In this case, while the fractional vortices on the poles give the nearby vortices the tendency to move downwards (or upwards), the other vortices also organize themselves in a complicated way to achieve the ground state. An example of this behavior can be seen in Fig. 5. We believe there must be a topological reason (yet to be understood) which relates these nonzero fractional values of the AB flux strength to the particular values of N.

In summary, we have presented an interesting example of an interplay between the geometry of a superconductor and the vortices on it resulting in the quantization of an AB flux with its strength given by a fraction of the fundamental flux quantum. We stress again that this quantization of an AB flux has a dynamical origin following from the minimization of the free energy for a vortex system. Since our analysis is based on the mean field theory, our results will be valid only in the very low temperature regime. As the temperature is raised, we expect the quantization effect will be weakened by thermal fluctuations and the AB flux will fluctuate around its minimum value. It would also be interesting to study possible relations of the present results to other physical systems exhibiting *fractional* vortices[6].

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TABLES

TABLE I. Fractional values (other than 1/2) of the Aharobov-Bohm flux quanta up to $N = 36$.

N	$a = -f/\Phi_0$
8	0.1305 ± 0.0005
14	0.2775 ± 0.0005
20	0.4365 ± 0.0005
22	0.1895 ± 0.0005
24	0.1915 ± 0.0005
30	0.3250 ± 0.0005
34	0.3545 ± 0.0005
36	0.3615 ± 0.0005

FIGURES
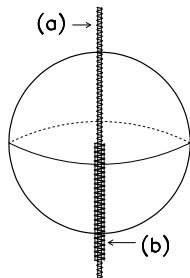
FIG. 1. A spherical superconductor penetrated by two solenoids: (b) provides an approximately radial magnetic field at the spherical surface and (a) carries an AB flux along the z-axis.
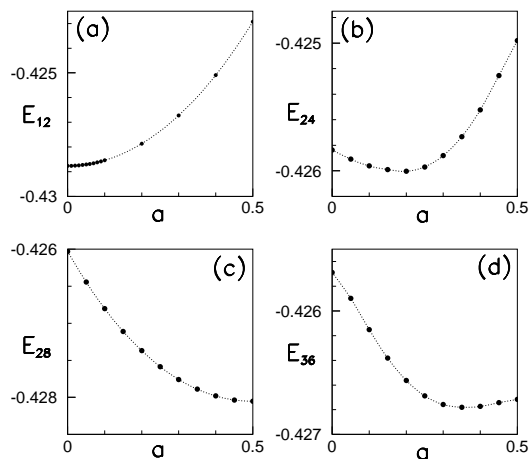
FIG. 2. The ground state energy per vortex as a function of the AB flux strength $a = -f/\Phi_0$ for various values of N; (a) $N = 12$, (b) $N = 24$, (c) $N = 28$, (d) $N = 36$. The dotted lines are guides to the eye
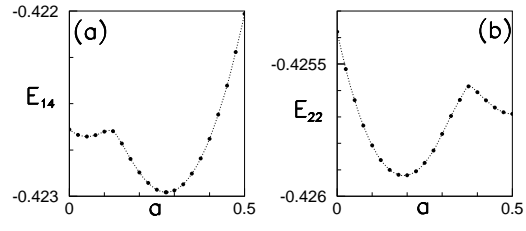
13

FIG. 3. The ground state energy per vortex as a function of the AB flux strength for (a) $N = 14$, and (b) $N = 22$.
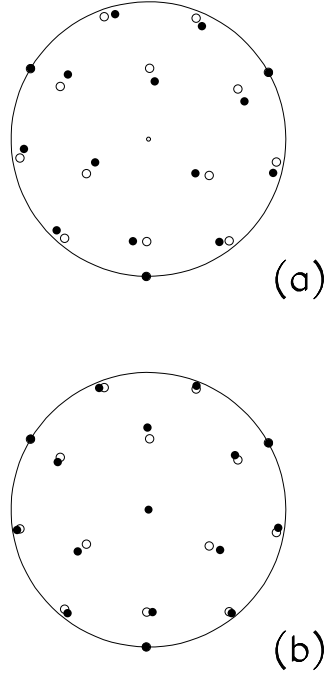
FIG. 4. The configuration of vortices for $N = 28$ and for $a = 0.05$ (filled circles) and for $a = 0.5$ (open circles). The case where $a = 0.5$ has a lower energy. Figures (a) and (b) describe the vortices on the northern and southern hemispheres respectively. The poles correspond to the centers of the circles, on which fractional vortices are represented by the smaller size of the symbols.
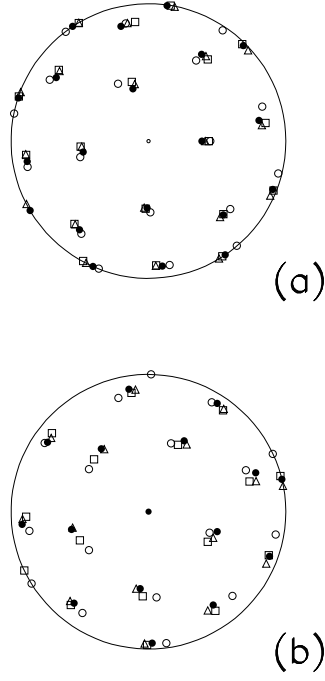
FIG. 5. The configuration of vortices for $N = 36$ and for $a = 0.15$ (filled circles), $a = 0.25$ (open triangles), $a = 0.35$ (open squares), and $a = 0.45$ (open circles). The case where $a = 0.35$ has the lowest energy. To compare, we have rotated the vortices until the nearest one to the north pole lies on the x-axis. (a) and (b) represent the vortices on the northern and southern hemispheres as in Fig. 2.